INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2008; **57**:1505–1526
Published online 27 November 2007 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/fld.1673

Optimal shape design for the time-dependent Navier–Stokes flow

Zhiming Gao[1,*,†], Yichen Ma[1] and Hongwei Zhuang[2]

[1] *School of Science, Xi'an Jiaotong University, Shaanxi 710049, People's Republic of China*
[2] *Engineering College of Armed Police Force, Shaanxi 710086, People's Republic of China*

SUMMARY

This paper is concerned with the problem of shape optimization of two-dimensional flows governed by the time-dependent Navier–Stokes equations. We derive the structures of shape gradients for time-dependent cost functionals by using the state derivative and its associated adjoint state. Finally, we apply a gradient-type algorithm to our problem, and numerical examples show that our theory is useful for practical purposes and the proposed algorithm is feasible in low Reynolds number flows. Copyright © 2007 John Wiley & Sons, Ltd.

Received 22 June 2007; Revised 5 October 2007; Accepted 14 October 2007

KEY WORDS: shape optimization; shape derivative; gradient algorithm; material derivative; time-dependent Navier–Stokes equations

1. INTRODUCTION

The problem of finding the optimal design of a system governed by the incompressible Navier–Stokes equations arises in many design problems in aerospace, automotive, hydraulic, ocean, structural, and wind engineering. Example applications include aerodynamic design of automotive vehicles, trains, low-speed aircraft, sails, and hydrodynamic design of ship hulls, turbomachinery, and offshore structures. In many cases, the flow equations do not admit steady-state solutions, and the optimization model must incorporate the time-dependent form of the Navier–Stokes equations.

Optimal shape design has received considerable attention already. Early works concerning the existence of solutions and differentiability of the quantity (such as, state, cost functional, etc.) with respect to shape deformation occupied most of the 1980s (see [1–5]), the stabilization of structures using a boundary variation technique has been fully addressed in [2, 4, 5]. However, a few

*Correspondence to: Zhiming Gao, School of Science, Xi'an Jiaotong University, P.O. Box 1844, Xi'an, Shaanxi 710049, People's Republic China.
†E-mail: ycma@mail.xjtu.edu.cn, dtgaozm@gmail.com

Contract/grant sponsor: National Natural Science Fund of China; contract/grant numbers: 10701001, 10671153

studies have considered the shape optimization of time-dependent flows (see [6–9]). Our concern in this article is on shape sensitivity analysis of time-dependent Navier–Stokes flow with small regularity data, and on deriving an efficient numerical approach for the solution of two-dimensional realizations of such problems.

In [**?**], we use the state derivative approach to solve a shape optimization problem governed by a Robin problem, and in [11, 12], we derive the expression of shape gradients for Stokes and Navier–Stokes optimization problem by this approach, respectively. In this paper, we use this approach and a weak implicit function theorem to derive the structures of shape gradients with respect to the shape of the variable domain for some given cost functionals in shape optimization problems for time-dependent Navier–Stokes flow with small regularity data.

This paper is organized as follows. In Section 2, we briefly recall the velocity method which is used for the characterization of the deformation of the shape of the domain and give the definitions of Eulerian derivative and shape derivative. We also give the description of the shape optimization problem for the time-dependent Navier–Stokes flow.

In Section 3, we employ the weak implicit function theorem to prove the existence of the weak Piola material derivative, and then give the description of the shape derivative. After that, we express the shape gradients of some typical cost functionals by introducing the corresponding linear adjoint state systems.

Finally in Section 4, we propose a gradient-type algorithm with some numerical examples to prove that our theory could be very useful for the practical purpose and the proposed algorithm is efficient in low Reynolds number flow.

2. PRELIMINARIES AND STATEMENT OF THE PROBLEM

2.1. Elements of the velocity method and notations

Domains Ω do not belong to a vector space and this requires the development of shape calculus to make sense of a 'derivative' or a 'gradient'. To realize it, there are about three types of techniques: Hadamard [13] normal variation method, the perturbation of the identity method by Simon [14] and the velocity method (see Céa [1], Delfour and Zolésio [2], and Zolésio [15]). We will use the velocity method that contains the others. For that purpose, we choose an open set D in $\mathbb{R}^N$ with the boundary ∂D piecewise C^k, and a velocity space $\mathbf{V}\in \mathrm{E}^k := \{\mathbf{V}\in C([0,\varepsilon]; \mathscr{D}^k(\bar{D},\mathbb{R}^N)) : \mathbf{V}\cdot\mathbf{n}_{\partial D} = 0 \text{ on } \partial D\}$, where ε is a small positive real number and $\mathscr{D}^k(\bar{D},\mathbb{R}^N)$ denotes the space of all k-times continuous differentiable functions with compact support contained in $\mathbb{R}^N$. The velocity field

$$\mathbf{V}(s)(x) = \mathbf{V}(s,x), \quad x\in D, \quad s\geqslant 0$$

belongs to $\mathscr{D}^k(\bar{D},\mathbb{R}^N)$ for each s. It can generate transformations

$$T_s(\mathbf{V})X = x(s,X), \quad s\geqslant 0, \quad X\in D$$

by the following dynamical system:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s}(s,X) &= \mathbf{V}(s,x(s)) \\ x(0,X) &= X \end{aligned} \tag{1}$$

DOI: 10.1002/fld

with the initial value X given. We denote the 'transformed domain' $T_s(\mathbf{V})(\Omega)$ by $\Omega_s(\mathbf{V})$ at $s \geqslant 0$, and also set $\partial\Omega_s := T_s(\partial\Omega)$.

There exists an interval $I = [0, \delta)$, $0 < \delta \leqslant \varepsilon$, and a one-to-one map T_s from $\bar{D}$ onto $\bar{D}$ such that:

(i) $T_0 = \mathrm{I}$;
(ii) $(s, x) \mapsto T_s(x)$ belongs to $C^1(I; C^k(D; D))$ with $T_s(\partial D) = \partial D$;
(iii) $(s, x) \mapsto T_s^{-1}(x)$ belongs to $C(I; C^k(D; D))$.

Such transformations are well studied in [2].

Furthermore, for sufficiently small $s > 0$, the Jacobian J_s is strictly positive:

$$J_s(x) := \det|\mathrm{D}T_s(x)| = \det \mathrm{D}T_s(x) > 0 \tag{2}$$

where $\mathrm{D}T_s(x)$ denotes the Jacobian matrix of the transformation T_s evaluated at a point $x \in D$ associated with the velocity field $\mathbf{V}$. We will also use the following notation: $\mathrm{D}T_s^{-1}(x)$ is the inverse of the matrix $\mathrm{D}T_s(x)$, ${}^*\mathrm{D}T_s^{-1}(x)$ is the transpose of the matrix $\mathrm{D}T_s^{-1}(x)$. These quantities also satisfy the following lemmas.

Lemma 2.1 (*Sokolowski and Zolésio* [5])
For any $\mathbf{V} \in E^k$, $\mathrm{D}T_s$ and J_s are invertible. Moreover, $\mathrm{D}T_s, \mathrm{D}T_s^{-1}$ are in $C^1([0, \varepsilon]; C^{k-1}(\bar{D}; \mathbb{R}^{N \times N}))$, and J_s, J_s^{-1} are in $C^1([0, \varepsilon]; C^{k-1}(\bar{D}; \mathbb{R}))$

Lemma 2.2 (*Sokolowski and Zolésio* [5])
$\boldsymbol{\varphi}$ is assumed to be a vector function in $C^1(D)^N$.

(1) $\mathrm{D}(T_s^{-1}) \circ T_s = \mathrm{D}T_s^{-1}$;
(2) $\mathrm{D}(\boldsymbol{\varphi} \circ T_s^{-1}) = (\mathrm{D}\boldsymbol{\varphi} \cdot \mathrm{D}T_s^{-1}) \circ T_s^{-1}$;
(3) $(\mathrm{D}\boldsymbol{\varphi}) \circ T_s = \mathrm{D}(\boldsymbol{\varphi} \circ T_s) \cdot \mathrm{D}T_s^{-1}$.

Now let $J(\Omega)$ be a real-valued functional associated with any regular domain Ω, we say that this functional has a *Eulerian derivative* at Ω in direction $\mathbf{V}$ if the limit

$$\lim_{s \searrow 0} \frac{J(\Omega_s) - J(\Omega)}{s} := \mathrm{d}J(\Omega; \mathbf{V})$$

exists.

Furthermore, if the map

$$\mathbf{V} \mapsto \mathrm{d}J(\Omega; \mathbf{V}) : \mathrm{E}^k \to \mathbb{R}$$

is linear and continuous, we say that J is *shape differentiable* at Ω. In the distributional sense we have

$$\mathrm{d}J(\Omega; \mathbf{V}) = \langle \nabla J, \mathbf{V} \rangle_{\mathscr{D}^k(\bar{D}, \mathbb{R}^N)' \times \mathscr{D}^k(\bar{D}, \mathbb{R}^N)} \tag{3}$$

When J has a Eulerian derivative, we say that ∇J is the *shape gradient* of J at Ω.

Before closing this subsection, we introduce the following functional spaces that will be used throughout this paper:

$$H(\operatorname{div}\Omega) := \{\mathbf{u} \in L^2(\Omega)^N : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

$$H_0^1(\operatorname{div}, \Omega) := \{\mathbf{u} \in H^1(\Omega)^N : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \mathbf{u}|_{\partial\Omega} = 0\}$$

DOI: 10.1002/fld

Given $T>0$, we introduce the notation $L^p(0,T;X)$ which denotes the space of L^p integrable functions f from $[0,T]$ into the Banach space X with the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|_X^p \,\mathrm{d}t\right)^{1/p}, \quad 1\leqslant p<+\infty$$

We also denote by $L^\infty(0,T;X)$ the space of essentially bounded functions f from $[0,T]$ into X, and equipped with the Banach norm

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_X$$

2.2. *Statement of the shape optimization problem*

In two dimensions, we consider a typical problem in which a solid body S with the boundary ∂S is located in an external flow. Since the flow is in an unbounded domain, we reduce the problem to a bounded domain D by introducing an artificial boundary ∂D on which we set the speed flow $\mathbf{y}=\mathbf{y}_\infty$. $\Omega:=D\backslash S$ is the effective domain with its boundary $\partial\Omega=\partial S\cup\partial D$. The state equations of the flow can be expressed by the Navier–Stokes equations in the non-dimensional form:

$$\begin{aligned}
\partial_t\mathbf{y}-\alpha\Delta\mathbf{y}+\mathrm{D}\mathbf{y}\cdot\mathbf{y}+\nabla p &= \mathbf{f} && \text{in } Q:=\Omega\times(0,T)\\
\operatorname{div}\mathbf{y} &= 0 && \text{in } Q\\
\mathbf{y} &= \mathbf{y}_\infty && \text{on } \partial D\times(0,T)\\
\mathbf{y} &= 0 && \text{on } \partial S\times(0,T)\\
\mathbf{y}(0) &= \mathbf{y}_0 && \text{in } \Omega\\
\int_\Omega p\,\mathrm{d}x &= 0 && \text{on } (0,T)\\
\int_{\partial D}\mathbf{y}_\infty\cdot\mathbf{n}\,\mathrm{d}s &= 0 && \text{on } (0,T)
\end{aligned} \tag{4}$$

where the last relation is needed in view of the incompressibility constraint $\operatorname{div}\mathbf{y}=0$; α stands for the inverse of the Reynolds number whenever the variables are appropriately non-dimensionalized, $\mathbf{y}$, p, and $\mathbf{f}$ are the velocity, pressure, and the given body force per unit mass, respectively.

Our goal is to optimize the shape of the boundary ∂S that minimizes a given cost functional J depending on the fluid state. The cost functional may represent a given objective related to specific characteristic features of the fluid flow (e.g. the deviation with respect to a given target velocity, the drag, the vorticity, etc.).

Hence, we are interested in solving the following minimization problem:

$$\min_{\Omega\in\mathcal{O}} J_1(\Omega)=\frac{1}{2}\int_0^T\int_\Omega |\mathbf{y}-\mathbf{y}_d|^2\,\mathrm{d}x\,\mathrm{d}t \tag{5}$$

or

$$\min_{\Omega\in\mathcal{O}} J_2(\Omega)=\frac{\alpha}{2}\int_0^T\int_\Omega |\operatorname{curl}\mathbf{y}|^2\,\mathrm{d}x\,\mathrm{d}t \tag{6}$$

where $\mathbf{y}$ is satisfied by the full Navier–Stokes system (4) and $\mathbf{y}_d$ is the target velocity given by the engineers. We also note that the boundary ∂D is fixed in our optimization problems and an example of the admissible set $\mathcal{O}$ is

$$\mathcal{O} := \left\{ \Omega \subset \mathbb{R}^N : \partial D \text{ is fixed}, \int_{\Omega} \mathrm{d}x = \text{constant} \right\}$$

In order to deal with the non-homogeneous Dirichlet boundary condition on ∂D, let the vectorial function $\mathbf{h}$ be the solution of

$$\begin{aligned} \operatorname{div} \mathbf{h} &= 0 && \text{in } \Omega \\ \mathbf{h} &= \mathbf{y}_\infty && \text{on } \partial D \\ \mathbf{h} &= 0 && \text{on } \partial S \end{aligned} \tag{7}$$

then we can choose an extension $\mathbf{h}$ with $\mathbf{h}=0$ in the body S.

Now we may look for a solution of the non-homogeneous Navier–Stokes equations in the form

$$\mathbf{y} = \mathbf{h} + \tilde{\mathbf{y}} \tag{8}$$

with $\tilde{\mathbf{y}}$ vanishing on the boundary of the domain Ω. Substituting (8) in system (4), we find the following equations for $\tilde{\mathbf{y}}$:

$$\begin{aligned} \partial_t \tilde{\mathbf{y}} - \alpha \Delta \tilde{\mathbf{y}} + \mathrm{D}\tilde{\mathbf{y}} \cdot \tilde{\mathbf{y}} + \mathrm{D}\tilde{\mathbf{y}} \cdot \mathbf{h} + \mathrm{D}\mathbf{h} \cdot \tilde{\mathbf{y}} + \nabla p &= \mathbf{F} && \text{in } Q \\ \operatorname{div} \tilde{\mathbf{y}} &= 0 && \text{in } Q \\ \tilde{\mathbf{y}} &= 0 && \text{on } \partial\Omega \times (0,T) \\ \tilde{\mathbf{y}}(0) &= \tilde{\mathbf{y}}_0 && \text{in } \Omega \end{aligned} \tag{9}$$

where $\mathbf{F} := \mathbf{f} + \alpha \Delta \mathbf{h} - \mathrm{D}\mathbf{h} \cdot \mathbf{h}$ and $\tilde{\mathbf{y}}_0 := \mathbf{y}_0 - \mathbf{h}$.

For the existence and uniqueness of the solution of the full Navier–Stokes system (9), we have the following results (see [16]).

Theorem 2.1

The domain Ω is supposed to be piecewise C^1. We assume that

$$\mathbf{f}, \partial_t \mathbf{f} \in L^2(0,T; H(\operatorname{div}, D)) \tag{10}$$

$$\mathbf{y}_0 \in H^2(D)^N \cap H_0^1(\operatorname{div}, D) \tag{11}$$

$$\mathbf{y}_\infty \in H^{3/2}(\partial D)^N \tag{12}$$

the solution of (4) is unique and satisfies

$$\tilde{\mathbf{y}}, \partial_t \tilde{\mathbf{y}} \in L^2(0,T; H_0^1(\operatorname{div}, \Omega)) \cap L^\infty(0,T; H(\operatorname{div}, \Omega))$$

Moreover, if Ω is of class C^2 and $\mathbf{f} \in L^\infty(0,T; H(\operatorname{div}, D))$, then the function $\tilde{\mathbf{y}} \in L^\infty(0,T; H^2(\Omega)^N)$.

DOI: 10.1002/fld

3. STATE DERIVATIVE APPROACH

In this section, we shall prove the main theorem using an approach based on the differentiability of the solution of the Navier–Stokes system (9) with respect to the variable domain. To begin with, we use the Piola transformation to bypass the divergence-free condition and then derive a weak material derivative by the weak implicit function theorem. Finally, we will derive the structure of the shape gradients of the cost functionals by introducing the associated adjoint state equations.

3.1. Piola material derivative

From now on, we assume that Ω is of class C^1 and (10)–(12) hold. Then we say that the function $\tilde{\mathbf{y}} \in L^2(0,T;H_0^1(\mathrm{div},\Omega))$ is called a weak solution of problem (9) if it satisfies

$$\langle e(\tilde{\mathbf{y}}), \mathbf{w}\rangle = \mathbf{0}, \quad \mathbf{w} \in L^2(0,T;H_0^1(\mathrm{div},\Omega)) \tag{13}$$

with $e(\tilde{\mathbf{y}}) := (e_1(\tilde{\mathbf{y}}), e_2(\tilde{\mathbf{y}}))$, $\mathbf{0} := (0,0)$, and

$$\langle e_1(\tilde{\mathbf{y}}), \mathbf{w}\rangle := \int_0^T \int_\Omega (\partial_t \tilde{\mathbf{y}} \cdot \mathbf{w} + \alpha \mathrm{D}\tilde{\mathbf{y}} : \mathrm{D}\mathbf{w} + \mathrm{D}\tilde{\mathbf{y}} \cdot \tilde{\mathbf{y}} \cdot \mathbf{w} + \mathrm{D}\tilde{\mathbf{y}} \cdot \mathbf{h} \cdot \mathbf{w} + \mathrm{D}\mathbf{h} \cdot \tilde{\mathbf{y}} \cdot \mathbf{w} - \mathbf{F} \cdot \mathbf{w})\,\mathrm{d}x\,\mathrm{d}t \tag{14}$$

$$\langle e_2(\tilde{\mathbf{y}}), \mathbf{w}\rangle := \int_\Omega (\tilde{\mathbf{y}}(0) - \tilde{\mathbf{y}}_0) \cdot \mathbf{w}(0)\,\mathrm{d}x \tag{15}$$

It must be considered that the divergence-free condition is variant with respect to the use of the transformation T_s during the derivation of the shape gradient for the cost functional. Therefore, we need to introduce the well-known Piola transformation that preserves the divergence-free condition.

Lemma 3.1 (*Boisgerault* [*17*])
The Piola transform

$$\Psi_s : H(\mathrm{div},\Omega) \mapsto H(\mathrm{div},\Omega_s)$$
$$\boldsymbol{\varphi} \mapsto ((J_s)^{-1}\mathrm{D}T_s \cdot \boldsymbol{\varphi}) \circ T_s^{-1}$$

is an isomorphism.

Now by the transformation T_s, we consider the solution $\tilde{\mathbf{y}}_s$ defined on $\Omega_s \times (0,T)$ of the perturbed weak formulation

$$\int_0^T \int_{\Omega_s} (\partial_t \tilde{\mathbf{y}}_s \cdot \mathbf{w}_s + \alpha \mathrm{D}\tilde{\mathbf{y}}_s : \mathrm{D}\mathbf{w}_s + \mathrm{D}\tilde{\mathbf{y}}_s \cdot \tilde{\mathbf{y}}_s \cdot \mathbf{w}_s + \mathrm{D}\tilde{\mathbf{y}}_s \cdot \mathbf{h} \cdot \mathbf{w}_s$$
$$+ \mathrm{D}\mathbf{h} \cdot \tilde{\mathbf{y}}_s \cdot \mathbf{w}_s - \mathbf{F} \cdot \mathbf{w}_s)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{16}$$

$$\int_{\Omega_s} (\tilde{\mathbf{y}}_s(0) - \tilde{\mathbf{y}}_0) \cdot \mathbf{w}_s(0)\,\mathrm{d}x = 0 \tag{17}$$

DOI: 10.1002/fld

for all $\mathbf{w}_s \in L^2(0,T;H_0^1(\mathrm{div},\Omega_s))$, and introduce $\tilde{\mathbf{y}}^s = \Psi_s^{-1}(\tilde{\mathbf{y}}_s)$, $\mathbf{w}^s = \Psi_s^{-1}(\mathbf{w}_s)$ defined on Q. Then we replace $\tilde{\mathbf{y}}_s$, $\mathbf{w}_s$ by $\Psi_s(\tilde{\mathbf{y}}^s)$, $\Psi_s(\mathbf{w}^s)$ in the weak system (16), (17):

$$\begin{aligned}\int_0^T \int_{\Omega_s} [\partial_t \Psi_s(\tilde{\mathbf{y}}^s)\cdot\Psi_s(\mathbf{w}^s) + \alpha \mathrm{D}(\Psi_s(\tilde{\mathbf{y}}^s)):\mathrm{D}(\Psi_s(\mathbf{w}^s))\\ + \mathrm{D}(\Psi_s(\tilde{\mathbf{y}}^s))\cdot\Psi_s(\tilde{\mathbf{y}}^s)\cdot\Psi_s(\mathbf{w}^s) + \mathrm{D}(\Psi_s(\tilde{\mathbf{y}}^s))\cdot\mathbf{h}\cdot\Psi_s(\mathbf{w}^s)\\ + \mathrm{D}\mathbf{h}\cdot\Psi_s(\tilde{\mathbf{y}}^s)\cdot\Psi_s(\mathbf{w}^s) - \mathbf{F}\cdot\Psi_s(\mathbf{w}^s)]\,\mathrm{d}x\,\mathrm{d}t = 0\end{aligned} \tag{18}$$

$$\int_{\Omega_s} [\Psi_s(\tilde{\mathbf{y}}^s(0)) - \tilde{\mathbf{y}}_0]\cdot\Psi_s(\mathbf{w}^s(0))\,\mathrm{d}x = 0 \tag{19}$$

for all $\mathbf{w}^s \in L^2(0,T;H_0^1(\mathrm{div},\Omega))$.

Using a back transport into Ω and employing Lemma 2.2, we obtain the following weak formulation:

$$\langle e(s,\tilde{\mathbf{y}}^s),\mathbf{w}^s\rangle = \mathbf{0} \quad \forall \mathbf{w}^s \in L^2(0,T;H_0^1(\mathrm{div},\Omega)) \tag{20}$$

with the notation $e := (e_1, e_2)$, where

$$\begin{aligned}\langle e_1(s,\mathbf{v}),\mathbf{w}\rangle := &\int_0^T \int_\Omega \partial_t(B(s)\mathbf{v})\cdot(\mathrm{D}T_s\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t\\ &+\alpha\int_0^T\int_\Omega \mathrm{D}(B(s)\mathbf{v}):[\mathrm{D}(B(s)\mathbf{w})\cdot A(s)]\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \mathrm{D}(B(s)\mathbf{v})\cdot\mathbf{v}\cdot(B(s)\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_\Omega \mathrm{D}(B(s)\mathbf{v})\cdot\mathbf{h}\cdot(B(s)\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \mathrm{D}(B(s)\mathbf{h})\cdot\mathbf{v}\cdot(B(s)\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_\Omega (\mathbf{F}\circ T_s)\cdot(\mathrm{D}T_s\cdot\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{21}$$

and

$$\langle e_2(s,\mathbf{v}),\mathbf{w}\rangle := \int_\Omega (B(s)\mathbf{v}(0) - \tilde{\mathbf{y}}_0\circ T_s)\cdot(\mathrm{D}T_s\mathbf{w}(0))\,\mathrm{d}x \tag{22}$$

and

$$A(s) := J_s \mathrm{D}T_s^{-1*}\mathrm{D}T_s^{-1}, \quad B(s)\boldsymbol{\tau} := J_s^{-1}\mathrm{D}T_s\cdot\boldsymbol{\tau}$$

Now we are interested in the differentiability of the mapping

$$s \mapsto \tilde{\mathbf{y}}^s = \Psi_s^{-1}(\tilde{\mathbf{y}}_s) : [0,\varepsilon] \mapsto L^2(0,T;H_0^1(\mathrm{div},\Omega))$$

where $\varepsilon > 0$ is sufficiently small and $\tilde{\mathbf{y}}^s$ is the solution of the weak formulation

$$\langle e(s,\mathbf{v}),\mathbf{w}\rangle = \mathbf{0} \quad \forall \mathbf{w} \in L^2(0,T;H_0^1(\mathrm{div},\Omega)) \tag{23}$$

DOI: 10.1002/fld

In order to prove the differentiability of $\tilde{\mathbf{y}}^s$ with respect to s in a neighborhood of $s=0$, there may be two approaches:

(i) Analysis of the differential quotient: $\lim_{s\to 0}(\tilde{\mathbf{y}}^s-\tilde{\mathbf{y}})/s$;
(ii) derivation of the local differentiability of the solution $\tilde{\mathbf{y}}$ associated with the implicit equation (13).

We use the second approach. Since $\mathbf{f}\in L^2(0,T;H(\mathrm{div},D))$, we deduce that $(\mathbf{f}\circ T_s-\mathbf{f})/s$ weakly converges to $\mathrm{D}\mathbf{f}\cdot\mathbf{V}$ in $L^2(0,T;H^{-1}(D)^N)$ as s goes to zero. Thus, we cannot use the classical implicit function theorem, since it requires strong differentiability results in H^{-1}. Hence, we introduce the following weak implicit function theorem.

Theorem 3.1 (*Zolésio* [*15*])
Let X, Y' be two Banach spaces, I an open bounded set in $\mathbb{R}$, and consider the map

$$(s,x)\mapsto e(s,x): I\times X\mapsto Y'$$

If the following hypotheses hold:

(i) $s\mapsto\langle e(s,x),y\rangle$ is continuously differentiable for any $y\in Y$ and $(s,x)\mapsto\langle\partial_s e(s,x),y\rangle$ is continuous;
(ii) there exists $u\in X$ such that $u\in C^{0,1}(I;X)$ and $e(s,u(s))=0$, $\forall s\in I$;
(iii) $x\mapsto e(s,x)$ is differentiable and $(s,x)\mapsto\partial_x e(s,x)$ is continuous;
(iv) there exists $s_0\in I$ such that $\partial_x e(s,x)|_{(s_0,x(s_0))}$ is an isomorphism from X to Y',

the mapping

$$s\mapsto u(s): I\mapsto X$$

is differentiable at $s=s_0$ for the weak topology in X and its weak derivative $\dot{u}(s)$ is the solution of

$$\langle\partial_x e(s_0,u(s_0))\cdot\dot{u}(s_0),y\rangle+\langle\partial_s e(s_0,u(s_0)),y\rangle=0\quad\forall y\in Y$$

We may now state the main theorem of this section.

Theorem 3.2
We assume that the domain Ω is piecewise C^1 and (10)–(12) hold, $\tilde{\mathbf{y}}\in L^2(0,T;H_0^1(\mathrm{div},\Omega))$ is the solution of the weak formulation (13). Then the weak *Piola material derivative* $\dot{\tilde{\mathbf{y}}}^P:=\partial_s(\tilde{\mathbf{y}}^s)|_{s=0}$ exists and is characterized by the following weak formulation:

$$\langle\partial_{\mathbf{v}}e(0,\mathbf{v})|_{\mathbf{v}=\tilde{\mathbf{y}}}\cdot\dot{\tilde{\mathbf{y}}}^P,\mathbf{w}\rangle+\langle\partial_s e(0,\tilde{\mathbf{y}}),\mathbf{w}\rangle=\mathbf{0}\quad\forall\mathbf{w}\in L^2(0,T;H_0^1(\mathrm{div},\Omega))\tag{24}$$

i.e.

$$\begin{aligned}&\int_0^T\int_\Omega[\partial_t\dot{\tilde{\mathbf{y}}}^P\cdot\mathbf{w}+\alpha\mathrm{D}\dot{\tilde{\mathbf{y}}}^P:\mathrm{D}\mathbf{w}+\mathrm{D}\dot{\tilde{\mathbf{y}}}^P\cdot\tilde{\mathbf{y}}\cdot\mathbf{w}+\mathrm{D}\tilde{\mathbf{y}}\cdot\dot{\tilde{\mathbf{y}}}^P\cdot\mathbf{w}+\mathrm{D}\dot{\tilde{\mathbf{y}}}^P\cdot\mathbf{h}\cdot\mathbf{w}+\mathrm{D}\mathbf{h}\cdot\dot{\tilde{\mathbf{y}}}^P\cdot\mathbf{w}]\,\mathrm{d}x\,\mathrm{d}t\\&=-\int_0^T\int_\Omega[\partial_t((\mathrm{D}\mathbf{V}-\mathrm{div}\,\mathbf{V})\tilde{\mathbf{y}})\cdot\mathbf{w}+\partial_t\tilde{\mathbf{y}}\cdot\mathrm{D}\mathbf{V}\cdot\mathbf{w}]\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\alpha\int_0^T\int_\Omega\mathrm{D}((\mathrm{D}\mathbf{V}-\mathrm{div}\,\mathbf{V})\tilde{\mathbf{y}}):\mathrm{D}\mathbf{w}\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&-\alpha\int_0^T\int_\Omega \mathrm{D}\tilde{\mathbf{y}}:[\mathrm{D}((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\mathbf{w})+\mathrm{D}\mathbf{w}\cdot(\operatorname{div}\mathbf{V}-\mathrm{D}\mathbf{V}-{}^*\mathrm{D}\mathbf{V})]\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega [\mathrm{D}((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})(\tilde{\mathbf{y}}+\mathbf{h}))\cdot\tilde{\mathbf{y}}\cdot\mathbf{w}+\mathrm{D}(\tilde{\mathbf{y}}+\mathbf{h})\cdot\tilde{\mathbf{y}}\cdot((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\mathbf{w})]\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega [\mathrm{D}((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\tilde{\mathbf{y}})\cdot\mathbf{h}\cdot\mathbf{w}-\mathrm{D}\tilde{\mathbf{y}}\cdot\mathbf{h}\cdot((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\mathbf{w})]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega ({}^*\mathrm{D}\mathbf{V}\cdot(\mathbf{f}+\alpha\Delta\mathbf{h}-\mathrm{D}\mathbf{h}\cdot\mathbf{h})+\mathrm{D}(\mathbf{f}+\alpha\Delta\mathbf{h}-\mathrm{D}\mathbf{h}\cdot\mathbf{h})\cdot\mathbf{V})\cdot\mathbf{w}\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega (\mathbf{f}+\alpha\Delta\mathbf{h}-\mathrm{D}\mathbf{h}\cdot\mathbf{h})\cdot(\mathrm{D}\mathbf{V}\cdot\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{25}
$$

and

$$\int_\Omega \dot{\tilde{\mathbf{y}}}^P(0)\cdot\mathbf{w}(0)\,\mathrm{d}x=-\int_\Omega[(\mathrm{D}\mathbf{V}+{}^*\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V}\,\mathrm{I})\cdot\tilde{\mathbf{y}}(0)-(\mathrm{D}\tilde{\mathbf{y}}_0\mathbf{V}+{}^*\mathrm{D}\mathbf{V}\tilde{\mathbf{y}}_0)]\cdot\mathbf{w}(0)\,\mathrm{d}x\tag{26}$$

Proof

In order to apply Theorem 3.1, we need to verify the four hypotheses of Theorem 3.1 for the mapping

$$(s,\mathbf{v})\mapsto e(s,\mathbf{v}):[0,\varepsilon]\times L^2(0,T;H_0^1(\operatorname{div},\Omega))\mapsto L^2(0,T;H_0^1(\operatorname{div},\Omega)')$$

To begin with, since Ω is of piecewise C^1, the mapping $T_s\in C^1([0,\varepsilon];C^1(D,D))$. Then by Lemma 2.1, the mapping

$$s\mapsto\langle e_i(s,\mathbf{v}),\mathbf{w}\rangle:[0,\varepsilon]\mapsto\mathbb{R}\quad(i=1,2)$$

is C^1 for any $\mathbf{v},\mathbf{w}\in L^2(0,T;H_0^1(\operatorname{div},\Omega))$. On the other hand, since $\mathbf{f}\in L^2(0,T;H(\operatorname{div},D))$, the mapping $s\mapsto\mathbf{f}\circ T_s$ is only weakly differentiable in H^{-1}; thus, the mapping $s\mapsto e_1(s,\mathbf{v})$ is weakly differentiable, and then $s\mapsto e(s,\mathbf{v})$ is weakly differentiable.

Since we have the following identities by simple calculation:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{D}T_s=(\mathrm{D}\mathbf{V}(s)\circ T_s)\mathrm{D}T_s\tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}J_s=(\operatorname{div}\mathbf{V}(s))\circ T_s\,J_s\tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{f}\circ T_s)=(\mathrm{D}\mathbf{f}\cdot\mathbf{V}(s))\circ T_s\tag{29}$$

the weak derivative of $e_i(s,\mathbf{v})$ $(i=1,2)$ can be expressed as

$$
\begin{aligned}
\langle\partial_s e_1(s,\mathbf{v}),\mathbf{w}\rangle=&\int_0^T\int_\Omega[\partial_t(B'(s)\mathbf{v})\cdot(\mathrm{D}T_s\mathbf{w})+\partial_t(B(s)\mathbf{v})\cdot(\mathrm{D}\mathbf{V}(s)\circ T_s)\cdot\mathrm{D}T_s\cdot\mathbf{w}]\,\mathrm{d}x\,\mathrm{d}t\\
&+\alpha\int_0^T\int_\Omega \mathrm{D}(B'(s)\mathbf{v}):[\mathrm{D}(B(s)\mathbf{w})\cdot A(s)]\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

DOI: 10.1002/fld

$$
\begin{aligned}
&+\alpha\int_0^T\int_\Omega \mathrm{D}(B(s)\mathbf{v}):[\mathrm{D}(B'(s)\mathbf{w})\cdot A(s)+\mathrm{D}(B(s)\mathbf{w})\cdot A'(s)]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega [\mathrm{D}(B'(s)\mathbf{v})\cdot\mathbf{v}\cdot(B(s)\mathbf{w})+\mathrm{D}(B(s)\mathbf{v})\cdot\mathbf{v}\cdot(B'(s)\mathbf{w})]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega [\mathrm{D}(B'(s)\mathbf{v})\cdot\mathbf{h}\cdot(B(s)\mathbf{w})+\mathrm{D}(B(s)\mathbf{v})\cdot\mathbf{h}\cdot(B'(s)\mathbf{w})]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega [\mathrm{D}(B'(s)\mathbf{h})\cdot\mathbf{v}\cdot(B(s)\mathbf{w})+\mathrm{D}(B(s)\mathbf{h})\cdot\mathbf{v}\cdot(B'(s)\mathbf{w})]\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega [{}^*\mathrm{D}\mathbf{V}(s)\cdot\mathbf{F}+\mathrm{D}\mathbf{F}\cdot\mathbf{V}(s)]\circ T_s\cdot(\mathrm{D}T_s\,\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega (\mathbf{F}\circ T_s)\cdot[(\mathrm{D}\mathbf{V}(s)\circ T_s)\cdot\mathrm{D}T_s\cdot\mathbf{w}]\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{30}
$$

and

$$
\begin{aligned}
\langle\partial_s e_2(s,\mathbf{v}),\mathbf{w}\rangle=\int_\Omega\{&[B'(s)\mathbf{v}(0)-(\mathrm{D}\tilde{\mathbf{y}}_0\cdot\mathbf{V}(s))\circ T_s]\cdot(\mathrm{D}T_s\mathbf{w}(0))\\
&+(B(s)\mathbf{v}(0)-\tilde{\mathbf{y}}_0\circ T_s)\cdot[(\mathrm{D}\mathbf{V}(s)\circ T_s)\cdot\mathrm{D}T_s\cdot\mathbf{w}(0)]\}\,\mathrm{d}x
\end{aligned}\tag{31}
$$

where

$$
B'(s)\boldsymbol{\tau}:=\frac{\partial}{\partial s}[B(s)\boldsymbol{\tau}]=[\mathrm{D}\mathbf{V}(s)\circ T_s-(\operatorname{div}\mathbf{V}(s)\circ T_s)\mathrm{I}]B(s)\boldsymbol{\tau}
$$

$$
A'(s):=\frac{\partial}{\partial s}A(s)=[\operatorname{div}\mathbf{V}(s)\circ T_s-\mathrm{D}T_s^{-1}\mathrm{D}\mathbf{V}(s)\circ T_s]A(s)-{}^*[\mathrm{D}T_s^{-1}\mathrm{D}\mathbf{V}(s)\circ T_s A(s)]
$$

Obviously, the mapping $(s,\mathbf{v})\mapsto\partial_s e(s,\mathbf{v})$ is continuous, and when we take $s=0$, we have

$$
B'(0)\boldsymbol{\tau}=(\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V}\mathrm{I})\cdot\boldsymbol{\tau}
$$

$$
A'(0)=\operatorname{div}\mathbf{V}\mathrm{I}-\mathrm{D}\mathbf{V}-{}^*\mathrm{D}\mathbf{V}
$$

and then

$$
\begin{aligned}
\langle\partial_s e_1(0,\mathbf{v}),\mathbf{w}\rangle=&\int_0^T\int_\Omega[\partial_t((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\mathbf{v})\cdot\mathbf{w}+\partial_t\mathbf{v}\cdot\mathrm{D}\mathbf{V}\cdot\mathbf{w}]\,\mathrm{d}x\,\mathrm{d}t\\
&+\alpha\int_0^T\int_\Omega \mathrm{D}((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\mathbf{v}):\mathrm{D}\mathbf{w}\,\mathrm{d}x\,\mathrm{d}t\\
&+\alpha\int_0^T\int_\Omega \mathrm{D}\mathbf{v}:[\mathrm{D}((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\mathbf{w})+\mathrm{D}\mathbf{w}\cdot(\operatorname{div}\mathbf{V}-\mathrm{D}\mathbf{V}-{}^*\mathrm{D}\mathbf{V})]\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega [\mathrm{D}((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})(\mathbf{v}+\mathbf{h}))\cdot\mathbf{v}\cdot\mathbf{w}+\mathrm{D}(\mathbf{v}+\mathbf{h})\cdot\mathbf{v}\cdot((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\mathbf{w})]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega [\mathrm{D}((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\mathbf{v})\cdot\mathbf{h}\cdot\mathbf{w}+\mathrm{D}\mathbf{v}\cdot\mathbf{h}\cdot((\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\mathbf{w})]\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega \mathbf{F}\cdot(\mathrm{D}\mathbf{V}\cdot\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega ({}^*\mathrm{D}\mathbf{V}\cdot\mathbf{F}+\mathrm{D}\mathbf{F}\cdot\mathbf{V})\cdot\mathbf{w}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{32}
$$

$$
\langle\partial_s e_2(0,\mathbf{v}),\mathbf{w}\rangle=\int_\Omega[(\mathrm{D}\mathbf{V}+{}^*\mathrm{D}\mathbf{V}-\operatorname{div}\mathbf{V})\cdot\mathbf{v}(0)\cdot\mathbf{w}(0)-(\mathrm{D}\tilde{\mathbf{y}}_0\mathbf{V}+{}^*\mathrm{D}\mathbf{V}\tilde{\mathbf{y}}_0)\cdot\mathbf{w}(0)]\,\mathrm{d}x \tag{33}
$$

To verify (ii), we follow the same steps described in Dziri [18] to find that the mapping $s\mapsto\tilde{\mathbf{y}}_s\circ T_s$ is Lipschitz continuous which is the direct consequence of the uniqueness of the solution of the Navier–Stokes system, i.e. Theorem 2.1.

It is easy to check that the mappings

$$
\mathbf{v}\mapsto e_1(s,\mathbf{v}):L^2(0,T;H_0^1(\operatorname{div},\Omega))\to L^2(0,T;H_0^1(\operatorname{div},\Omega)')
$$

$$
\mathbf{v}\mapsto e_2(s,\mathbf{v}):H_0^1(\operatorname{div},\Omega)\to H_0^1(\operatorname{div},\Omega)'
$$

are differentiable, and the derivatives of $e_i(s,\mathbf{v})$ with respect to $\mathbf{v}$ in the direction $\delta\mathbf{v}$ are

$$
\begin{aligned}
\langle\partial_v e_1(s,\mathbf{v})\cdot\delta\mathbf{v},\mathbf{w}\rangle=&\int_0^T\int_\Omega \partial_t(B(s)\delta\mathbf{v})\cdot(\mathrm{D}T_s\mathbf{w})\,\mathrm{d}x\,\mathrm{d}t\\
&+\alpha\int_0^T\int_\Omega \mathrm{D}(B(s)\delta\mathbf{v}):[\mathrm{D}(B(s)\mathbf{w})\cdot A(s)]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega[\mathrm{D}(B(s)\delta\mathbf{v})\cdot\mathbf{v}\cdot(B(s)\mathbf{w})+\mathrm{D}(B(s)\mathbf{v})\cdot\delta\mathbf{v}\cdot(B(s)\mathbf{w})]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega[\mathrm{D}(B(s)\delta\mathbf{v})\cdot\mathbf{h}\cdot(B(s)\mathbf{w})+\mathrm{D}(B(s)\mathbf{h})\cdot\delta\mathbf{v}\cdot(B(s)\mathbf{w})]\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{34}
$$

and

$$
\langle\partial_v e_2(s,\mathbf{v})\cdot\delta\mathbf{v},\mathbf{w}\rangle=\int_\Omega(B(s)\delta\mathbf{v}(0))\cdot(\mathrm{D}T_s\cdot\mathbf{w}(0))\,\mathrm{d}x \tag{35}
$$

The continuity of $(s,\mathbf{v})\mapsto\partial_v e_i(s,\mathbf{v})$ is easy to check. Moreover,

$$
\begin{aligned}
\langle\partial_v e_1(0,\mathbf{v})\cdot\delta\mathbf{v},\mathbf{w}\rangle=&\int_0^T\int_\Omega[\partial_t(\delta\mathbf{v})\cdot\mathbf{w}+\alpha\mathrm{D}(\delta\mathbf{v}):\mathrm{D}\mathbf{w}+\mathrm{D}(\delta\mathbf{v})\cdot\mathbf{v}\cdot\mathbf{w}\\
&\times\mathrm{D}\mathbf{v}\cdot\delta\mathbf{v}\cdot\mathbf{w}+\mathrm{D}(\delta\mathbf{v})\cdot\mathbf{h}\cdot\mathbf{w}+\mathrm{D}\mathbf{h}\cdot\delta\mathbf{v}\cdot\mathbf{w}]\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{36}
$$

$$
\langle\partial_v e_2(0,\mathbf{v})\cdot\delta\mathbf{v},\mathbf{w}\rangle=\int_\Omega\delta\mathbf{v}(0)\cdot\mathbf{w}(0)\,\mathrm{d}x \tag{37}
$$

DOI: 10.1002/fld

Furthermore, $\delta\mathbf{v} \to \partial_v e(0, \mathbf{v}) \cdot \delta\mathbf{v}$ is an isomorphism that follows from the uniqueness and existence of the Navier–Stokes system, i.e. Theorem 2.1. Indeed, we assume that $\tilde{\mathbf{y}}_1, \tilde{\mathbf{y}}_2$ are two solutions of the Navier–Stokes system (9), and $\tilde{\mathbf{y}}_i$ $(i=1,2)$ satisfies the weak formulation (13). It is obvious that $\hat{\mathbf{y}} = \tilde{\mathbf{y}}_1 - \tilde{\mathbf{y}}_2$ satisfies

$$\int_0^T \int_\Omega [\partial_t \hat{\mathbf{y}} \cdot \mathbf{w} + \alpha \mathrm{D}\hat{\mathbf{y}} : \mathrm{D}\mathbf{w} + \mathrm{D}\hat{\mathbf{y}} \cdot \mathbf{h} \cdot \mathbf{w} + \mathrm{D}\mathbf{h} \cdot \hat{\mathbf{y}} \cdot \mathbf{w} + \mathrm{D}\hat{\mathbf{y}} \cdot \tilde{\mathbf{y}}_1 \cdot \mathbf{w} + \mathrm{D}\tilde{\mathbf{y}}_2 \cdot \hat{\mathbf{y}} \cdot \mathbf{w}] \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{38}$$

and

$$\int_\Omega \hat{\mathbf{y}}(0) \cdot \mathbf{w}(0) \, \mathrm{d}x = 0 \tag{39}$$

Now let $\mathbf{w} = \hat{\mathbf{y}}$, we can follow the proof of the unique solvability of the unsteady Navier–Stokes equations (see Temam [16]) and obtain

$$|\hat{\mathbf{y}}(t)|^2 \leqslant 0 \quad \forall t \in [0, T]$$

Thus $\tilde{\mathbf{y}}_1 = \tilde{\mathbf{y}}_2$. Similar *a priori* estimates hold for $\delta\mathbf{v}$ and the uniqueness of the solution of system (36), (37) is obtained.

Finally, all the hypotheses are satisfied by (20), we can apply Theorem 3.1 to (20) and then use (32), (33), (36), and (37) to obtain (25) and (26). □

3.2. Shape derivative

In this subsection, we will characterize the shape derivative $\tilde{\mathbf{y}}'$, i.e. the derivative of the state $\tilde{\mathbf{y}}$ with respect to the shape of the variable domain.

Theorem 3.3

Under the assumption of Theorem 2.1 and moreover assume that Ω is of class C^2, $\tilde{\mathbf{y}} \in L^\infty(0, T; H^2(\Omega)^N \cap H_0^1(\mathrm{div}, \Omega))$ solves the weak formulation (13) and $\tilde{\mathbf{y}}_s$ solves the perturbed weak formulation (16) (17) in $\Omega_s \times (0, T)$, then the *shape derivative*

$$\tilde{\mathbf{y}}' := \lim_{s \to 0} \frac{\tilde{\mathbf{y}}_s - \tilde{\mathbf{y}}}{s}$$

exists and is characterized as the solution of

$$\begin{aligned}
\partial_t \tilde{\mathbf{y}}' - \alpha \Delta \tilde{\mathbf{y}}' + \mathrm{D}\tilde{\mathbf{y}}' \cdot \tilde{\mathbf{y}} + \mathrm{D}\tilde{\mathbf{y}} \cdot \tilde{\mathbf{y}}' + \mathrm{D}\tilde{\mathbf{y}}' \cdot \mathbf{h} + \mathrm{D}\mathbf{h} \cdot \tilde{\mathbf{y}}' + \nabla p' &= 0 && \text{in } Q \\
\mathrm{div}\, \tilde{\mathbf{y}}' &= 0 && \text{in } Q \\
\tilde{\mathbf{y}}' &= -(\mathrm{D}\tilde{\mathbf{y}} \cdot \mathbf{n}) \mathbf{V}_n && \text{on } \partial S \times (0, T) \\
\tilde{\mathbf{y}}' &= 0 && \text{on } \partial D \times (0, T) \\
\tilde{\mathbf{y}}'(0) &= 0 && \text{in } \Omega
\end{aligned} \tag{40}$$

Proof

Since Ω is of class C^2 and $\mathbf{V} \in \mathrm{E}^2$, Ω_s has the same regularity than Ω for any $s \in (0, \varepsilon)$, then $\tilde{\mathbf{y}}_s \in L^\infty(0, T; H^2(\Omega_s)^N)$ satisfies the following weak formulation:

$$\int_0^T \int_{\Omega_s} (\alpha \mathrm{D}\tilde{\mathbf{y}}_s : \mathrm{D}\mathbf{w} + \mathrm{D}\tilde{\mathbf{y}}_s \cdot \tilde{\mathbf{y}}_s \cdot \mathbf{w} + \mathrm{D}\tilde{\mathbf{y}}_s \cdot \mathbf{h} \cdot \mathbf{w} + \mathrm{D}\mathbf{h} \cdot \tilde{\mathbf{y}}_s \cdot \mathbf{w} - \mathbf{F} \cdot \mathbf{w}) \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{41}$$

$$\int_{\Omega_s} \tilde{\mathbf{y}}_s(0)\cdot\mathbf{w}(0)\,\mathrm{d}x = 0 \tag{42}$$

for any $\mathbf{w}\in L^2(0,T;H_0^1(\mathrm{div},\Omega_s))$. Moreover, we have $\partial_t\tilde{\mathbf{y}}_s \in L^2(0,T;H_0^1(\mathrm{div},\Omega))$.

To begin with, we introduce the following Hadamard formula (see [2, 5]):

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\Omega_s} g(s,x)\,\mathrm{d}x = \int_{\Omega_s}\frac{\partial g}{\partial s}(s,x)\,\mathrm{d}x + \int_{\partial\Omega_s} g(s,x)\mathbf{V}\cdot\mathbf{n}_s\,\mathrm{d}\Gamma_s \tag{43}$$

for a sufficiently smooth functional $g:[0,\tau]\times\mathbb{R}^N\to\mathbb{R}$.

Now we set a function $\boldsymbol{\varphi}\in\mathscr{D}(Q)^{N+1}$ and $\mathrm{div}\,\boldsymbol{\varphi}(x,t)=0$ in Ω for a.e. $t\in(0,T)$. Obviously when s is sufficiently small, $\boldsymbol{\varphi}(t)$ belongs to the Sobolev space $H_0^1(\mathrm{div},\Omega_s)\cap H^2(\Omega_s)^N$ for a.e. $t\in(0,T)$. Hence, we can use (43) to differentiate (41), (42) with $\mathbf{w}=\boldsymbol{\varphi}$:

$$\int_0^T\int_\Omega(\partial_t\tilde{\mathbf{y}}'+\alpha\mathrm{D}\tilde{\mathbf{y}}':\mathrm{D}\boldsymbol{\varphi}+\mathrm{D}\tilde{\mathbf{y}}'\cdot\tilde{\mathbf{y}}\cdot\boldsymbol{\varphi}+\mathrm{D}\tilde{\mathbf{y}}\cdot\tilde{\mathbf{y}}'\cdot\boldsymbol{\varphi}+\mathrm{D}\tilde{\mathbf{y}}'\cdot\mathbf{h}\cdot\boldsymbol{\varphi}+\mathrm{D}\mathbf{h}\cdot\tilde{\mathbf{y}}'\cdot\boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^T\int_{\partial\Omega}(\alpha\mathrm{D}\tilde{\mathbf{y}}:\mathrm{D}\boldsymbol{\varphi}+\mathrm{D}\tilde{\mathbf{y}}\cdot\tilde{\mathbf{y}}\cdot\boldsymbol{\varphi}+\mathrm{D}\tilde{\mathbf{y}}\cdot\mathbf{h}\cdot\boldsymbol{\varphi}+\mathrm{D}\mathbf{h}\cdot\tilde{\mathbf{y}}\cdot\boldsymbol{\varphi}-\mathbf{F}\cdot\boldsymbol{\varphi})\mathbf{V}_n\,\mathrm{d}s\,\mathrm{d}t=0$$

$$\int_\Omega\tilde{\mathbf{y}}'(0)\cdot\boldsymbol{\varphi}(0)\,\mathrm{d}x+\int_{\partial\Omega}\tilde{\mathbf{y}}_s(0)\cdot\boldsymbol{\varphi}(0)\mathbf{V}_n\,\mathrm{d}s=0$$

Since $\boldsymbol{\varphi}$ has a compact support, the boundary integrals vanish. Using integration by parts, we obtain

$$\int_0^T\int_\Omega(\partial_t\tilde{\mathbf{y}}'-\alpha\Delta\tilde{\mathbf{y}}'+\mathrm{D}\tilde{\mathbf{y}}'\cdot\tilde{\mathbf{y}}+\mathrm{D}\tilde{\mathbf{y}}\cdot\tilde{\mathbf{y}}'+\mathrm{D}\tilde{\mathbf{y}}'\cdot\mathbf{h}+\mathrm{D}\mathbf{h}\cdot\tilde{\mathbf{y}}')\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=0 \tag{44}$$

and

$$\int_\Omega\tilde{\mathbf{y}}'(0)\cdot\boldsymbol{\varphi}(0)\,\mathrm{d}x=0 \tag{45}$$

Then there exists some distribution p' such that

$$\partial_t\tilde{\mathbf{y}}'-\alpha\Delta\tilde{\mathbf{y}}'+\mathrm{D}\tilde{\mathbf{y}}'\cdot\tilde{\mathbf{y}}+\mathrm{D}\tilde{\mathbf{y}}\cdot\tilde{\mathbf{y}}'+\mathrm{D}\tilde{\mathbf{y}}'\cdot\mathbf{h}+\mathrm{D}\mathbf{h}\cdot\tilde{\mathbf{y}}'=-\nabla p'$$

in the distributional sense in Q and $\tilde{\mathbf{y}}'(0)=0$ in Ω since $\boldsymbol{\varphi}(0)$ is arbitrary.

Now we recall that for each sufficient small s, $\Psi_s^{-1}(\tilde{\mathbf{y}}_s)$ belongs to the Sobolev space $H_0^1(\mathrm{div},\Omega)$, then we can deduce that its material derivative vanishes on the boundary ∂S. Thus, we obtain the shape derivative of $\tilde{\mathbf{y}}$ at the boundary ∂S:

$$\tilde{\mathbf{y}}'=-\mathrm{D}\tilde{\mathbf{y}}\cdot\mathbf{V}\quad\text{on }\partial S\times(0,T)$$

Since $\tilde{\mathbf{y}}|_{\partial S\times(0,T)}=0$, we have $\mathrm{D}\tilde{\mathbf{y}}|_{\partial S\times(0,T)}=\mathrm{D}\tilde{\mathbf{y}}\cdot\mathbf{n}^*\mathbf{n}$, and then

$$\tilde{\mathbf{y}}'=-(\mathrm{D}\tilde{\mathbf{y}}\cdot\mathbf{n})\mathbf{V}_n\quad\text{on }\partial S\times(0,T)$$

Since ∂D is fixed, we obtain $\tilde{\mathbf{y}}'=0$ on the boundary $\partial D\times(0,T)$. □

DOI: 10.1002/fld

The shape derivative $\mathbf{y}'$ of the solution $\mathbf{y}$ of the original Navier–Stokes system (4) is given by $\tilde{\mathbf{y}}'=\mathbf{y}'$, then we obtain the following corollary by substituting $\tilde{\mathbf{y}}'=\mathbf{y}'$ and $\tilde{\mathbf{y}}=\mathbf{y}-\mathbf{h}$ into (40).

Corollary 3.1
The shape derivative $\mathbf{y}'$ of the solution $\mathbf{y}$ of (4) exists and satisfies the following system:

$$\begin{aligned}
\partial_t \mathbf{y}' - \alpha\Delta\mathbf{y}' + \mathrm{D}\mathbf{y}'\cdot\mathbf{y} + \mathrm{D}\mathbf{y}\cdot\mathbf{y}' + \nabla p' &= 0 && \text{in } Q\\
\operatorname{div}\mathbf{y}' &= 0 && \text{in } Q\\
\mathbf{y}' &= (-\mathrm{D}\mathbf{y}\cdot\mathbf{n})\mathbf{V}_n && \text{on } \partial S\times(0,T)\\
\mathbf{y}' &= 0 && \text{on } \partial D\times(0,T)\\
\mathbf{y}'(0) &= 0 && \text{in } \Omega
\end{aligned} \tag{46}$$

3.3. Adjoint state system and gradients of the cost functionals

This subsection is devoted to the computation of the shape gradients for the cost functionals $J_1(\Omega)$ and $J_2(\Omega)$ by the adjoint method.

For the cost functional $J_1(\Omega)=\int_0^T\int_\Omega \frac{1}{2}|\mathbf{y}-\mathbf{y}_d|^2\,\mathrm{d}x\,\mathrm{d}t$, we have the following.

Theorem 3.4
Let Ω be of class C^2, $\mathbf{y}_d\in L^\infty(0,T;L^2(D)^N)$, and $\mathbf{V}\in \mathrm{E}^2$; the shape gradient ∇J_1 of the cost functional $J_1(\Omega)$ can be expressed as

$$\nabla J_1 = [\tfrac{1}{2}(\mathbf{y}-\mathbf{y}_d)^2 + \alpha(\mathrm{D}\mathbf{y}\cdot\mathbf{n})\cdot(\mathrm{D}\mathbf{v}\cdot\mathbf{n})]\mathbf{n} \tag{47}$$

where the adjoint state $\mathbf{v}$ satisfies the following linear adjoint system:

$$\begin{aligned}
-\partial_t\mathbf{v} - \alpha\Delta\mathbf{v} - \mathrm{D}\mathbf{v}\cdot\mathbf{y} + {}^*\mathrm{D}\mathbf{y}\cdot\mathbf{v} + \nabla q &= \mathbf{y}-\mathbf{y}_d && \text{in } Q\\
\operatorname{div}\mathbf{v} &= 0 && \text{in } Q\\
\mathbf{v} &= 0 && \text{on } \partial\Omega\times(0,T)\\
\mathbf{v}(T) &= 0 && \text{in } \Omega
\end{aligned} \tag{48}$$

Proof
Since $J_1(\Omega)$ is differentiable with respect to $\mathbf{y}$, and the state $\mathbf{y}$ is shape differentiable with respect to s, i.e. the shape derivative $\mathbf{y}'$ exists, we obtain the Eulerian derivative of $J_1(\Omega)$ with respect to s:

$$\mathrm{d}J_1(\Omega;\mathbf{V}) = \int_0^T\int_\Omega (\mathbf{y}-\mathbf{y}_d)\cdot\mathbf{y}'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega}\frac{1}{2}|\mathbf{y}-\mathbf{y}_d|^2\mathbf{V}_n\,\mathrm{d}s\,\mathrm{d}t \tag{49}$$

by the Hadamard formula (43).

DOI: 10.1002/fld

By the Green formula, we have the following identity:

$$\begin{aligned}
&\int_0^T \int_\Omega [(\partial_t \mathbf{y}' - \alpha\Delta\mathbf{y}' + \mathrm{D}\mathbf{y}'\cdot\mathbf{y} + \mathrm{D}\mathbf{y}\cdot\mathbf{y}' + \nabla p')\cdot\mathbf{w} - \operatorname{div}\mathbf{y}'\pi]\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega [(-\partial_t \mathbf{w} - \alpha\Delta\mathbf{w} - \mathrm{D}\mathbf{w}\cdot\mathbf{y} + {}^*\mathrm{D}\mathbf{y}\cdot\mathbf{w} + \nabla\pi)\cdot\mathbf{y}' - p'\operatorname{div}\mathbf{w}]\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^T \int_{\partial\Omega} (\mathbf{y}'\cdot\mathbf{w})(\mathbf{y}\cdot\mathbf{n})\,\mathrm{d}s\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} (\alpha\mathrm{D}\mathbf{w}\cdot\mathbf{n} - \pi\mathbf{n})\cdot\mathbf{y}'\,\mathrm{d}s\,\mathrm{d}t \\
&\qquad + \int_0^T \int_{\partial\Omega} (p'\mathbf{n} - \alpha\mathrm{D}\mathbf{y}'\mathbf{n})\cdot\mathbf{w}\,\mathrm{d}s\,\mathrm{d}t + \int_\Omega (\mathbf{y}'(T)\cdot\mathbf{w}(T) - \mathbf{y}'(0)\cdot\mathbf{w}(0))\,\mathrm{d}x
\end{aligned} \tag{50}$$

Now we define $(\mathbf{v}, q)$ to be the solution of (48), use (46) and set $(\mathbf{w}, \pi) = (\mathbf{v}, q)$ in (50) to obtain

$$\int_0^T \int_\Omega (\mathbf{y} - \mathbf{y}_d)\cdot\mathbf{y}'\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\partial S} (\alpha\mathrm{D}\mathbf{v}\cdot\mathbf{n} - q\mathbf{n})\cdot\mathbf{y}'\,\mathrm{d}s\,\mathrm{d}t \tag{51}$$

Since $\mathbf{y}' = (-\mathrm{D}\mathbf{y}\cdot\mathbf{n})\mathbf{V}_n$ on the boundary ∂S and $\operatorname{div}\mathbf{y}' = 0$ in Ω, we obtain the Eulerian derivative of $J_1(\Omega)$ from (49):

$$\mathrm{d}J_1(\Omega; \mathbf{V}) = \int_0^T \int_{\partial S} \left[\frac{1}{2}|\mathbf{y} - \mathbf{y}_d|^2 + \alpha(\mathrm{D}\mathbf{y}\cdot\mathbf{n})\cdot(\mathrm{D}\mathbf{v}\cdot\mathbf{n})\right]\mathbf{V}_n\,\mathrm{d}s\,\mathrm{d}t \tag{52}$$

Since the mapping $\mathbf{V} \mapsto \mathrm{d}J_1(\Omega; \mathbf{V})$ is linear and continuous, we obtain the expression (47) for the shape gradient ∇J_1 by (3). □

For another typical cost functional $J_2(\Omega) = (\alpha/2)\int_0^T \int_\Omega |\operatorname{curl}\mathbf{y}|^2\,\mathrm{d}x\,\mathrm{d}t$, we have the following theorem.

Theorem 3.5
Let Ω be of class C^2 and $\mathbf{V} \in \mathrm{E}^2$, the cost functional $J_2(\Omega)$ possesses the shape gradient ∇J_2 which can be expressed as

$$\nabla J_2 = \alpha[\tfrac{1}{2}|\operatorname{curl}\mathbf{y}|^2 + (\mathrm{D}\mathbf{y}\cdot\mathbf{n})\cdot(\mathrm{D}\mathbf{v}\cdot\mathbf{n} - \operatorname{curl}\mathbf{y}\wedge\mathbf{n})]\mathbf{n} \tag{53}$$

where the adjoint state $\mathbf{v}$ satisfies the following linear adjoint system:

$$\begin{aligned}
-\partial_t\mathbf{v} - \alpha\Delta\mathbf{v} - \mathrm{D}\mathbf{v}\cdot\mathbf{y} + {}^*\mathrm{D}\mathbf{y}\cdot\mathbf{v} + \nabla q &= -\alpha\Delta\mathbf{y} && \text{in } Q \\
\operatorname{div}\mathbf{v} &= 0 && \text{in } Q \\
\mathbf{v} &= 0 && \text{on } \partial\Omega\times(0,T) \\
\mathbf{v}(T) &= 0 && \text{in } \Omega
\end{aligned} \tag{54}$$

DOI: 10.1002/fld

Proof
The proof is similar to that of Theorem 3.4. Using the Hadamard formula (43) for the cost functional J_2, we obtain the Eulerian derivative:

$$\mathrm{d}J_2(\Omega;\mathbf{V}) = \alpha \int_0^T \int_\Omega \operatorname{curl}\mathbf{y}\cdot\operatorname{curl}\mathbf{y}'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} \frac{\alpha}{2}|\operatorname{curl}\mathbf{y}|^2\mathbf{V}_n\,\mathrm{d}s\,\mathrm{d}t \tag{55}$$

Then, we define $(\mathbf{v}, q)$ to be the solution of (54), use (46) and set $(\mathbf{w}, \pi) = (\mathbf{v}, q)$ in (50) to obtain

$$\alpha \int_0^T \int_\Omega \Delta\mathbf{y}\cdot\mathbf{y}'\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\partial S} \alpha(\mathrm{D}\mathbf{v}\cdot\mathbf{n})\cdot\mathbf{y}'\,\mathrm{d}s\,\mathrm{d}t \tag{56}$$

Applying the following vectorial Green formula:

$$\int_\Omega (\boldsymbol{\varphi}\cdot\Delta\boldsymbol{\psi} + \operatorname{curl}\boldsymbol{\varphi}\cdot\operatorname{curl}\boldsymbol{\psi} + \operatorname{div}\boldsymbol{\varphi}\operatorname{div}\boldsymbol{\psi})\,\mathrm{d}x$$

$$= \int_{\partial\Omega} (\boldsymbol{\varphi}\cdot(\operatorname{curl}\boldsymbol{\psi}\wedge\mathbf{n}) + \boldsymbol{\varphi}\cdot\mathbf{n}\operatorname{div}\boldsymbol{\psi})\,\mathrm{d}s$$

for vector functions $\mathbf{y}$ and $\mathbf{y}'$, we obtain

$$\int_0^T \int_\Omega (\operatorname{curl}\mathbf{y}\cdot\operatorname{curl}\mathbf{y}' + \Delta\mathbf{y}\cdot\mathbf{y}')\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\partial S} (\operatorname{curl}\mathbf{y}\wedge\mathbf{n})\cdot\mathbf{y}'\,\mathrm{d}s\,\mathrm{d}t \tag{57}$$

Combining (55) and (56) with (57), we obtain the Eulerian derivative:

$$\mathrm{d}J_2(\Omega;\mathbf{V}) = \int_0^T \int_{\partial S} \alpha\left[\frac{1}{2}|\operatorname{curl}\mathbf{y}|^2 + (\mathrm{D}(\mathbf{y}-\mathbf{g})\cdot\mathbf{n})\cdot(\mathrm{D}\mathbf{v}\cdot\mathbf{n} - \operatorname{curl}\mathbf{y}\wedge\mathbf{n})\right]\mathbf{V}_n\,\mathrm{d}s\,\mathrm{d}t$$

Finally, we arrive at the expression (53) for the shape gradient ∇J_2. □

4. GRADIENT ALGORITHM AND NUMERICAL SIMULATION

In this section, we will give a gradient-type algorithm and some numerical examples in two dimensions to prove that our previous methods could be very useful and efficient for the numerical implementation of the shape optimization problems for the unsteady Navier–Stokes flow. For the sake of simplicity, we only consider the cost functional $J(\Omega) = \int_0^T \int_\Omega |\mathbf{y} - \mathbf{y}_d|^2\,\mathrm{d}x\,\mathrm{d}t$.

4.1. A gradient-type algorithm

As we have just seen, the general form of the Eulerian derivative is

$$\mathrm{d}J(\Omega;\mathbf{V}) = \int_0^T \int_{\partial S} \nabla J\cdot\mathbf{V}\,\mathrm{d}s\,\mathrm{d}t$$

DOI: 10.1002/fld

where ∇J denotes the shape gradient of the cost functional J. Ignoring regularization, a descent direction is found by defining

$$\mathbf{V} = -h_k \nabla J \tag{58}$$

and then we can update the shape Ω as

$$\Omega_k = (\mathrm{I} + h_k \mathbf{V})\Omega \tag{59}$$

where h_k is a descent step at kth iteration.

There are also other choices for the definition of the descent direction. Since the gradient of the functional has necessarily less regularity than the parameter, an iterative scheme similar to the method of descent deteriorates the regularity of the optimized parameter. We need to project or smooth the variation into $H^1(\Omega)^2$. Hence, the method used in this paper is to change the scalar product with respect to which we compute a descent direction, for instance, $H^1(\Omega)^2$. In this case, the descent direction is the unique element $\mathbf{d} \in H^1(\Omega)^2$ such that at a fixed time $t \in [0, T]$ and for every $\mathbf{V} \in H^1(\Omega)^2$:

$$\int_\Omega \mathrm{D}\mathbf{d} : \mathrm{D}\mathbf{V}\, \mathrm{d}x = -\int_{\partial S} \nabla J \cdot \mathbf{V}\, \mathrm{d}s \tag{60}$$

The computation of $\mathbf{d}$ can also be interpreted as a regularization of the shape gradient, and the choice of $H^1(\Omega)^2$ as space of variations is more dictated by technical considerations rather than theoretical ones.

The resulting algorithm can be summarized as follows:

(1) Choose an initial shape Ω_0, i.e. choose an initial shape of ∂S since ∂D is fixed in our problem;
(2) Compute the state system (4) and adjoint state system (48); then we can evaluate the descent direction $\mathbf{d}_k$ by using (60) with $\Omega = \Omega_k$;
(3) Set $\Omega_{k+1} = (\mathrm{Id} + h_k \mathbf{d}_k)\Omega_k$, where h_k is a small positive real number.

The choice of the descent step h_k is not an easy task. If too big, the algorithm is unstable; if too small, the rate of convergence is insignificant. In order to refresh h_k, we compare h_k with h_{k-1}. If $(\mathbf{d}_k, \mathbf{d}_{k-1})_{H^1}$ is negative, we should reduce the step; on the other hand, if $\mathbf{d}_k$ and $\mathbf{d}_{k-1}$ are very close, we increase the step. In addition, if reversed triangles appear when moving the mesh, we also need to reduce the step.

In our algorithm, we do not choose any stopping criterion. A classical stopping criterion is to find whether the shape gradients ∇J in some suitable norm is small enough. However, since we use the continuous shape gradients, it is hopeless for us to expect very small gradient norm because of numerical discretization errors. Instead, we fix the number of iterations. If it is too small, we can restart it with the previous final shape as the initial shape.

4.2. Numerical examples

To illustrate the theory, we wish to solve the following minimization problem:

$$\min_\Omega \frac{1}{2} \int_0^1 \int_\Omega |\mathbf{y} - \mathbf{y}_d|^2\, \mathrm{d}x\, \mathrm{d}t \tag{61}$$

DOI: 10.1002/fld

subject to

$$
\begin{aligned}
\partial_t \mathbf{y} - \alpha\Delta\mathbf{y} + \mathrm{D}\mathbf{y}\cdot\mathbf{y} + \nabla p &= \mathbf{f} && \text{in } \Omega\times(0,1)\\
\operatorname{div}\mathbf{y} &= 0 && \text{in } \Omega\times(0,1)\\
\mathbf{y} &= 0 && \text{on } \partial S\times(0,1)\\
\mathbf{y} &= \mathbf{y}_\infty && \text{on } \partial D\times(0,1)\\
\mathbf{y}(0) &= 0 && \text{in } \Omega
\end{aligned}
\tag{62}
$$

where $D:=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leqslant 0.64\}$, and the shape of the body S is to be optimized. We choose the velocity $\mathbf{y}_\infty=(0.15y,-0.15x)^{\mathrm{T}}$ and the body force $\mathbf{f}=(f_1,f_2)^{\mathrm{T}}$:

$$
\begin{aligned}
f_1 = &-\frac{45x}{31\sqrt{x^2+y^2}} + \frac{\alpha t y(15x^2+15y^2-1)}{5(x^2+y^2)^{3/2}}\\
&+\frac{1}{25}t^2 x\left(-46-25x^2-25y^2-\frac{1}{x^2+y^2}+\frac{12}{\sqrt{x^2+y^2}}+60\sqrt{x^2+y^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
f_2 = &-\frac{45y}{31\sqrt{x^2+y^2}} - \frac{\alpha t x(15x^2+15y^2-1)}{5(x^2+y^2)^{3/2}}\\
&+\frac{1}{25}t^2 y\left(-46-25x^2-25y^2-\frac{1}{x^2+y^2}+\frac{12}{\sqrt{x^2+y^2}}+60\sqrt{x^2+y^2}\right)
\end{aligned}
$$

The target velocity $\mathbf{y}_d$ is determined by the data $\mathbf{f},\mathbf{y}_\infty$, and the target shape of the domain Ω. Our aim is to recover the shape of S which is a circle: $\partial S=\{(x,y): x^2+y^2=0.04\}$.

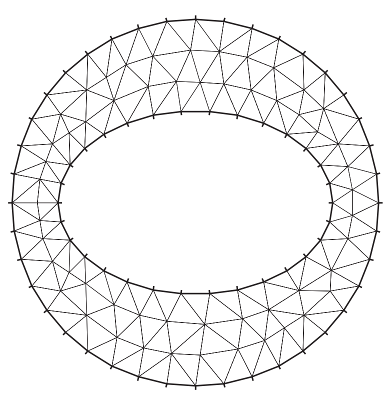

Figure 1. Initial mesh with 125 nodes.

DOI: 10.1002/fld

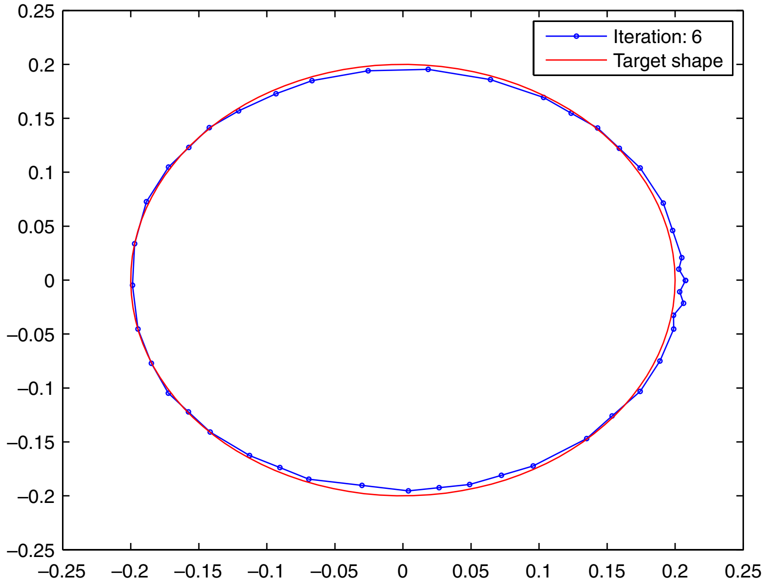

Figure 2. $\alpha = 0.1$ and CPU time: 124.531 s.

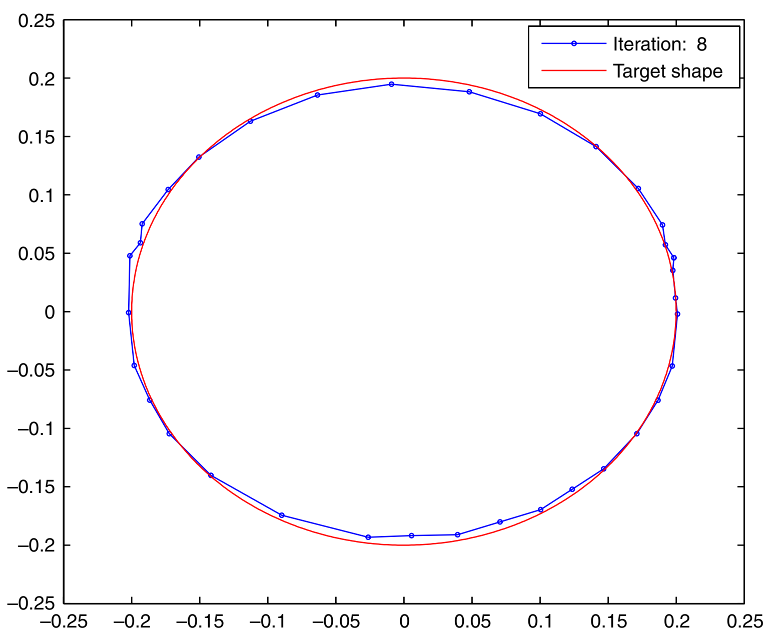

Figure 3. $\alpha = 0.01$ and CPU time: 120.125 s.

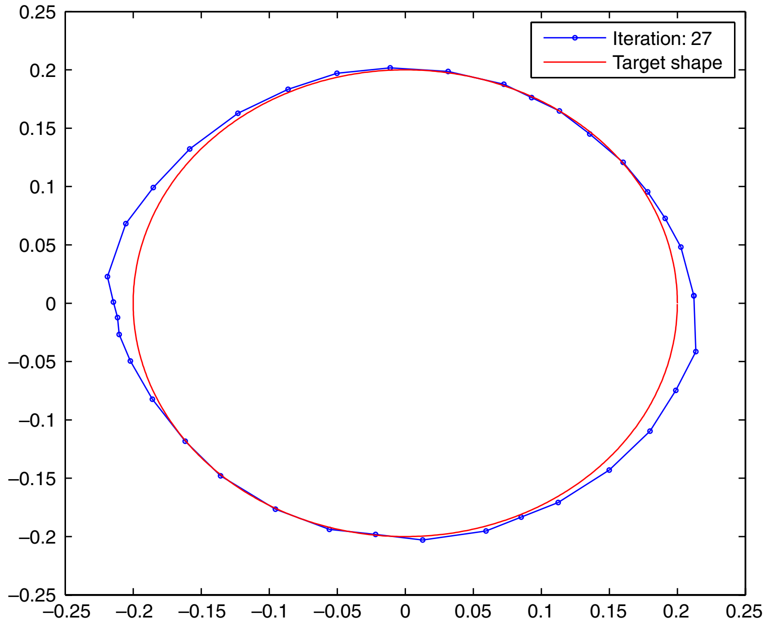

Figure 4. $\alpha = 0.001$ and CPU time: 622.813 s.

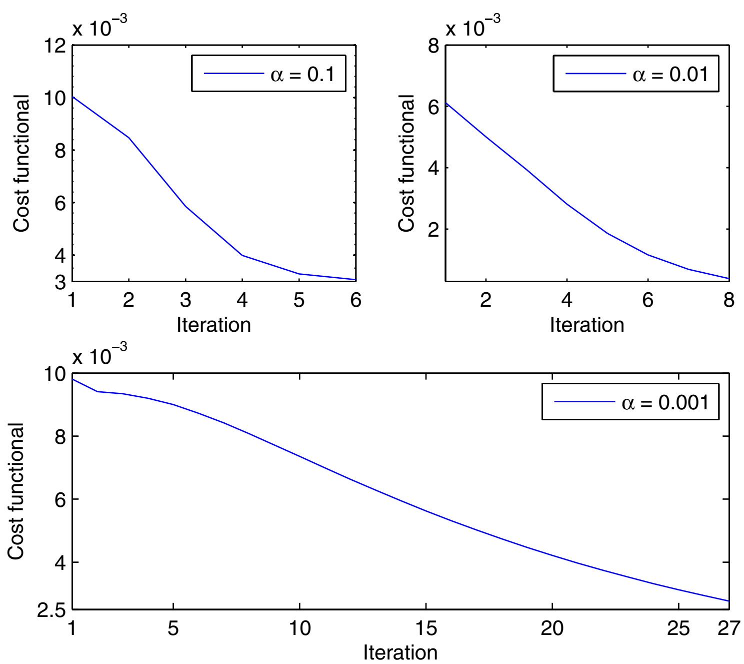

Figure 5. Convergence history for $\alpha = 0.1, 0.01$, and 0.001.

DOI: 10.1002/fld

The Navier–Stokes system (4) and the adjoint system (48) are discretized by using a mixed finite element method. Time discretization is effected using the backward Euler method and we assume that the time interval $[0,1]$ is divided into equal intervals of duration $\Delta t=0.05$. Spatial discretization is effected using the Taylor–Hood pair [19] of finite element spaces on a triangular mesh, i.e. the finite element spaces are chosen to be continuous piecewise quadratic polynomials for the velocity and continuous piecewise linear polynomials for the pressure. Our numerical solutions are obtained under FreeFem++ [20] and we run the program on a home PC.

We choose the initial shape of S to be elliptic: $\{(x,y): x^2/9+y^2/4=\frac{1}{25}\}$, and the initial finite element mesh was shown in Figure 1.

Figures 2–4 give the comparison between the target shape with iterated shape for the viscosity coefficients $\alpha=0.1, 0.01$, and 0.001, respectively. In case of $\alpha=0.1, 0.01$, we have fine results in Figures 2 and 3. Unfortunately, we cannot a nice reconstruction for $\alpha=0.001$ as in Figure 4.

Figure 5 represents the fast convergence of the cost functional for the various viscosity coefficients $\alpha=0.1, 0.01$, and 0.001.

5. CONCLUSION

In this paper, the shape optimization in the two-dimensional time-dependent Navier–Stokes flow has been presented. We employed the weak implicit function theorem to obtain the existence of the weak Piola material derivative; then we gave the description of the shape derivative. Hence, we derived the structures of shape gradients for some time-dependent cost functionals by introducing the associated adjoint state system. A gradient-type algorithm is effectively used for the minimization problem in various Reynolds number flows. Further research is necessary on efficient implementations for very large Reynolds numbers and real problems in the industry.

ACKNOWLEDGEMENTS

The authors would like to thank the referee for his (her) criticism, valuable comments which helped to improve this paper. This work was supported by the National Natural Science Fund of China under grant numbers 10701001, 10671153 for Z. M. Gao and Y. C. Ma.

REFERENCES

1. Céa J. Problems of shape optimal design. In *Optimization of Distributed Parameter Structures*, vol. II, Haug EJ, Céa J (eds). Sijthoff and Noordhoff: Alphen aan denRijn, The Netherlands, 1981; 1005–1048.
2. Delfour MC, Zolésio J-P. Shapes and geometries: analysis, differential Calculus, and optimization. *Advance in Design and Control*. SIAM: Philadelphia, PA, 2002.
3. Pironneau O. On optimum design in fluid mechanics. *Journal of Fluid Mechanics* 1974; **64**(part I):97–110.
4. Pironneau O. *Optimal Shape Design for Elliptic Systems*. Springer: Berlin, 1984.
5. Sokolowski J, Zolésio J-P. *Introduction to Shape Optimization*: *Shape Sensitivity Analysis*. Springer: Berlin, 1992.
6. Dziri R *et al*. Navier–Stokes dynamical shape control: from state derivative to Min–Max principle. *INRIA., N 4610*, October 2002.
7. He B *et al*. Computational strategies for shape optimization of time-dependent Navier–Stokes flows. *Technical Report CMU-CML-97-102*, Computational Mechanics Laboratory, Carnegie Mellon University, June 1997.
8. Mohammadi B, Pironneau O. *Applied Shape Optimization for Fluids*. Clardendon Press: Oxford, 2001.
9. Yagi H, Kawahara M. Shape optimization of a body located in low Reynolds number flow. *International Journal for Numerical Methods in Fluids* 2005; **48**:819–833.

DOI: 10.1002/fld

10. Gao ZM, Ma YC. Shape sensitivity analysis for a Robin problem via minimax differentiability. *Applied Mathematics and Computation* 2006; **181**(2):1090–1105.
11. Gao ZM, Ma YC, Zhuang HW. Shape optimization for Stokes flow. *Applied Numerical Mathematics*, 2007; DOI: 10.1016/j.apnum. 2007.03.001.
12. Gao ZM, Ma YC, Zhuang HW. Shape optimization for Navier–Stokes flow. *Inverse Problems in Science and Engineering* 2007; in press.
13. Hadamard J. Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. *Mémoire des savants étrangers*, vol. 33, 1907.
14. Simon J. Differentiation with respect to the domain in boundary value problems. *Numerical Functional Analysis and Optimzation* 1980; **2**:649–687.
15. Zolésio J-P. Identification de domaines par déformation. *Thèse de doctorat d'état*, Université de Nice, France, 1979.
16. Temam R. *Navier Stokes Equations*, *Theory and Numerical Analysis*. AMS Chelsea edition. American Mathematical Society: Rhode Island, 2001.
17. Boisgerault S. Optimisation de forme: systemes nonlineaires et mecanique des fluides. *Ph.D. Thesis*, Ecole des Mines de Paris - Informatique Temps reel, Robotique, Automatique, 2000.
18. Dziri R. Problemes de frontiere libre en fluides visqueux. *Ph.D. Thesis*, Ecole des Mines de Paris-Informatique Temps reel, Robotique, Automatique, 1995.
19. Hood P, Taylor G. Numerical solution of the Navier–Stokes equations using the finite element technique. *Computers and Fluids* 1973; **1**:1–28.
20. Hecht F, Pironneau O, Le Hyaric A, Ohtsuka K. *FreeFem++ Manual*, 2006. Available from: http://www.freefem.org.

DOI: 10.1002/fld